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On rank invariance of generalized Schwarz–Pick–Potapov block matrices of matrix-valued Carathéodory functions ☆

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Abstract

In view of a multiple Nevanlinna–Pick interpolation problem, we study the rank of generalized Schwarz–Pick–Potapov block matrices of matrix-valued Carathéodory functions. Those matrices are determined by the values of a Carathéodory function and the values of its derivatives up to a certain order. We derive statements on rank invariance of such generalized Schwarz–Pick–Potapov block matrices. These results are applied to describe the case of exactly one solution for the finite multiple Nevanlinna–Pick interpolation problem and to discuss matrix-valued Carathéodory functions with the highest degree of degeneracy.

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1. Introduction

This paper deals with matrix-valued Carathéodory functions in the open unit disk $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. Throughout, let q be a positive integer. By a $q \times q$ *Carathéodory function* $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ we mean a function which is holomorphic in $\mathbb{D}$ and which has a nonnegative Hermitian real part $\frac{1}{2}(\Omega(w) + [\Omega(w)]^*)$ for each $w \in \mathbb{D}$. We will use the notation $\mathscr{C}_q(\mathbb{D})$ to designate the set of all $q \times q$ Carathéodory functions (in $\mathbb{D}$).

The treatment of Nevanlinna–Pick interpolation problems for matrix-valued functions has led to the consideration of certain block matrices which are now called Schwarz–Pick–Potapov block matrices (cf. [9, Chapter 2]). Namely, it is known that such a problem has a solution if and only if the corresponding Schwarz–Pick–Potapov block matrix is nonnegative Hermitian. Potapov (see [20,11]) found several matrix inequalities which can be considered as far-reaching generalizations of the classical lemma of Schwarz and its reformulation by Pick. These inequalities form the basis of Potapov's "Method of Fundamental Matrix Inequality", which turned out to be a powerful tool to treat matrix versions of interpolation and moment problems (see, e.g., [17,8]).

In this paper, we consider a *multiple Nevanlinna–Pick interpolation problem* for Carathéodory functions as follows:

Problem (MNP). Let q be a positive integer, m be a nonnegative integer, and $l_0, l_1, \ldots, l_m$ be nonnegative integers as well, let $(\beta_k)_{k=0}^{m}$ be a sequence of distinct points belonging to $\mathbb{D}$, and let $\mathbf{\Omega}_{kt}$ be a complex $q \times q$ matrix for each $t \in \{0, 1, \ldots, l_k\}$ with $k \in \{0, 1, \ldots, m\}$. Find necessary and sufficient conditions for the existence of a $q \times q$ Carathéodory function Ω satisfying

$$\frac{1}{t!}\Omega^{(t)}(\beta_k) = \mathbf{\Omega}_{kt}, \quad t \in \{0, 1, \ldots, l_k\}, \quad k \in \{0, 1, \ldots, m\}. \tag{1}$$

Furthermore, describe the set of all $\Omega \in \mathscr{C}_q(\mathbb{D})$ fulfilling (1).

A multiple point interpolation problem is a problem where not only values for the function itself, but also for its derivatives up to a certain order are prescribed. Hence, the Problem (MNP) can be conceived as a generalization of the Carathéodory coefficient problem on the one hand (see [4,1]) and on the other hand of the classical Nevanlinna–Pick problem (see [19,18]).

There exists several approaches to the solution of such a problem and many other generalizations of the problem (see, e.g., [12,6,22,10,2,13,23,7]). A well-known fact is that a finite interpolation problem of Nevanlinna–Pick type can be frequently reduced, in a suitable way, to the study of a truncated trigonometric moment problem. In particular, Chen and Hu show in [5] a one-to-one correspondence between the finite multiple Nevanlinna–Pick matrix interpolation problem in the Carathéodory class and the truncated trigonometric moment problem and use this correspondence to solve the Problem (MNP) in both the nondegenerate and degenerate cases simultaneously.

In [15] the first steps towards constructing a matrix generalization of the theory of orthogonal rational functions created by Bultheel, González-Vera, Hendriksen, and Njåstad (see, e.g., [3]) are taken. There we obtained several statements on rank invariance of various Gramian matrices. These results are the starting point of the present paper. From them we derive corresponding results on rank invariance of generalized Schwarz–Pick–Potapov block matrices of matrix-valued Carathéodory functions which play a key role in [5] by the investigations on the Problem (MNP). In particular, we will see that this rank concept can be treated both on the basis of Taylor coefficients and on the basis of the values of the underlying Carathéodory functions.

A brief synopsis is as follows. In Section 2 we define some basic notations which we will use in the sequel. In Section 3 we introduce the generalized Schwarz–Pick–Potapov block matrices of the first kind corresponding to the given data in the Problem (MNP) and we recall the well-known criteria for the existence of a function $\Omega \in \mathscr{C}_q(\mathbb{D})$ satisfying (1). Moreover, as a main part of the paper we study in Section 3 the rank of the generalized Schwarz–Pick–Potapov block matrices of the first kind and we apply the statements on rank invariance obtained in the process to describe the case of exactly one solution of the Problem (MNP). The proof of these main results are given in Section 4. Here, an essential tool are right and left $\mathbb{C}^{q\times q}$-modules of rational matrix-valued functions. Furthermore, the considerations in Section 5 imply that the right and left versions which occur decoupled in the generalized Schwarz–Pick–Potapov block matrices of the first kind can be combined to so-called generalized Schwarz–Pick–Potapov block matrices of the second kind. We get a similar result on rank invariance for such kind of matrices. At the end of this paper, in Section 6 we present some conclusions out of the main results. In particular, the attention will be paid to the full rank case and, moreover, we discuss the special case of matrix-valued Carathéodory functions with atomic Riesz–Herglotz measure.

2. Notation

We will use $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ to denote the sets of all complex numbers, of all real numbers, of all integers, of all positive integers, and of all nonnegative integers, respectively. If $m \in \mathbb{N}_0$ and if $n \in \mathbb{N}_0$ or $n = +\infty$, then $\mathbb{N}_{m,n}$ stands for the set of all integers k which satisfy $m \leqslant k \leqslant n$. The symbol $\mathbb{T}$ stands for the unit circle, $\mathbb{D}$ for its interior, and $\mathbb{C}_0$ denotes the extended complex plane, i.e. $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, and $\mathbb{C}_0 := \mathbb{C} \cup \{\infty\}$.

Let $m, n \in \mathbb{N}_0$ with $n \geq m$. We use $\mathscr{T}_{m,n}$ to denote all sequences $(\alpha_j)_{j=m}^n$ of complex numbers fulfilling $\alpha_j\overline{\alpha_k} \neq 1$ for all integers $j, k \in \mathbb{N}_{m,n}$. The set of all sequences $(\alpha_j)_{j=m}^n \in \mathscr{T}_{m,n}$ of distinct points will be designated by $\mathscr{T}_{m,n}^{\#}$.

If $\mathfrak{X}$ is a nonempty set, we let $\mathfrak{X}^{q\times q}$ be the set of all $q \times q$ matrices each entry of which belongs to $\mathfrak{X}$. The notation $\mathbf{0}_{q\times q}$ stands for the null matrix that belongs to $\mathbb{C}^{q\times q}$ and the identity matrix which belongs to $\mathbb{C}^{q\times q}$ is designated by $\mathbf{I}_q$. If

$\mathbf{A} \in \mathbb{C}^{q\times q}$, let $\mathfrak{Re}\mathbf{A}$ and $\mathfrak{Im}\mathbf{A}$ be the real part of $\mathbf{A}$ and the imaginary part of $\mathbf{A}$, i.e. $\mathfrak{Re}\mathbf{A} := \frac{1}{2}(\mathbf{A}+\mathbf{A}^*)$ and $\mathfrak{Im}\mathbf{A} := \frac{1}{2\mathrm{i}}(\mathbf{A}-\mathbf{A}^*)$.

If Ω is a $q \times q$ matrix-valued function defined on $\mathbb{D}$ then let $\widehat{\Omega} : \mathbb{C}_0 \setminus \mathbb{T} \to \mathbb{C}^{q\times q}$ be defined by

$$\widehat{\Omega}(w) := \begin{cases} \Omega(w) & \text{if } w \in \mathbb{D}, \\ -\left[\Omega\left(\frac{1}{\overline{w}}\right)\right]^* & \text{if } w \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T}), \\ -[\Omega(0)]^* & \text{if } w = \infty. \end{cases} \tag{2}$$

If Ω is a complex $q \times q$ matrix-valued function which is holomorphic in a neighborhood of the point $v = 0$ then, for each $n \in \mathbb{N}_0$, let $\mathbf{S}_n^{(\Omega)}$ be the block Toeplitz matrix given by

$$\mathbf{S}_n^{(\Omega)} := \begin{pmatrix} \mathbf{A}_0 & \mathbf{0}_{q\times q} & \mathbf{0}_{q\times q} & \cdots & \mathbf{0}_{q\times q} \\ \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0}_{q\times q} & \cdots & \mathbf{0}_{q\times q} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{A}_{n-1} & \mathbf{A}_{n-2} & \cdots & \mathbf{A}_0 & \mathbf{0}_{q\times q} \\ \mathbf{A}_n & \mathbf{A}_{n-1} & \cdots & \mathbf{A}_1 & \mathbf{A}_0 \end{pmatrix},$$

where $\Omega(w) = \sum_{t=0}^{\infty} \mathbf{A}_t w^t$ is the Taylor expansion at $v = 0$ of Ω. For a given sequence $(\mathbf{A}_t)_{t=0}^n$ of complex $q \times q$ matrices we will also set

$$\mathbf{S}_{(\mathbf{A}_t)_{t=0}^n} := \begin{pmatrix} \mathbf{A}_0 & \mathbf{0}_{q\times q} & \mathbf{0}_{q\times q} & \cdots & \mathbf{0}_{q\times q} \\ \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0}_{q\times q} & \cdots & \mathbf{0}_{q\times q} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{A}_{n-1} & \mathbf{A}_{n-2} & \cdots & \mathbf{A}_0 & \mathbf{0}_{q\times q} \\ \mathbf{A}_n & \mathbf{A}_{n-1} & \cdots & \mathbf{A}_1 & \mathbf{A}_0 \end{pmatrix}. \tag{3}$$

3. Generalized Schwarz–Pick–Potapov matrices of the first kind

Starting from the Problem (MNP), let $m \in \mathbb{N}_0$, let $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$, let $(\beta_k)_{k=0}^m$ be a sequence of distinct points belonging to $\mathbb{D}$, and let $\mathbf{\Omega}_{kt}$ be a complex $q \times q$ matrix for $t \in \mathbb{N}_{0,l_k}$ and $k \in \mathbb{N}_{0,m}$. We denote this data set by Δ:

$$\Delta := \left((\beta_k)_{k=0}^m; \mathbf{\Omega}_{kt},\ t \in \{0, 1, \ldots, l_k\},\ k \in \{0, 1, \ldots, m\}\right) \tag{4}$$

and we put

$$n := m + \sum_{k=0}^{m} l_k. \tag{5}$$

Furthermore, we introduce the complex $(l_j + 1)q \times (l_k + 1)q$-matrices

$$\mathbf{P}_{jk} := (\mathbf{p}_{st}^{(jk)})_{\substack{s=0,1,\ldots,l_j \\ t=0,1,\ldots,l_k}}$$

$$\left(\text{respectively, } \widetilde{\mathbf{P}}_{jk} := \left([\mathbf{p}_{st}^{(jk)}]^*\right)_{\substack{s=0,1,\ldots,l_j\\ t=0,1,\ldots,l_k}}\right), \quad j,k \in \mathbb{N}_{0,m},$$

with entries

$$\begin{aligned}\mathbf{p}_{st}^{(jk)} :=& \sum_{h=0}^{s}\sum_{r=0}^{\min\{t,h\}} \frac{(h+t-r)!}{(t-r)!r!(h-r)!}\frac{\beta_j^{t-r}\overline{\beta_k}^{h-r}}{(1-\beta_j\overline{\beta_k})^{h+t-r+1}}\mathbf{\Omega}_{j,s-h}\\ &+\sum_{h=0}^{t}\sum_{r=0}^{\min\{s,h\}} \frac{(h+s-r)!}{(s-r)!r!(h-r)!}\frac{\beta_j^{h-r}\overline{\beta_k}^{s-r}}{(1-\beta_j\overline{\beta_k})^{h+s-r+1}}[\mathbf{\Omega}_{k,t-h}]^*,\\ & s\in\mathbb{N}_{0,l_j},\ t\in\mathbb{N}_{0,l_k}\end{aligned} \tag{6}$$

and the *generalized Schwarz–Pick–Potapov block matrix of the first kind* by

$$\mathbf{P}_\Delta := (\mathbf{P}_{jk})_{j,k=0}^m \quad \left(\text{respectively, } \widetilde{\mathbf{P}}_\Delta := (\widetilde{\mathbf{P}}_{jk})_{j,k=0}^m\right)$$

(which is of size $(n+1)q \times (n+1)q$). If we define the $(l_j+1)q \times (l_k+1)q$-matrices

$$\mathbf{W}_{jk} := \left(\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s\partial w^t}\left[\frac{1}{1-vw}\mathbf{I}_q\right]_{\substack{v=\beta_j\\ w=\overline{\beta_k}}}\right)_{\substack{s=0,1,\ldots,l_j\\ t=0,1,\ldots,l_k}}, \quad j,k\in\mathbb{N}_{0,m},$$

and, by using this notation and (3), the $(n+1)q \times (n+1)q$ matrices

$$\mathbf{W} := \left(\mathbf{W}_{jk}\right)_{j,k=0}^m, \quad \mathbf{S} := \operatorname{diag}\left(\mathbf{S}_{(\mathbf{\Omega}_{0t})_{t=0}^{l_0}}, \mathbf{S}_{(\mathbf{\Omega}_{1t})_{t=0}^{l_1}}, \ldots, \mathbf{S}_{(\mathbf{\Omega}_{mt})_{t=0}^{l_m}}\right),$$

then a straightforward calculation leads to the identities

$$\mathbf{P}_{jk} = \mathbf{S}_{(\mathbf{\Omega}_{js})_{s=0}^{l_j}}\mathbf{W}_{jk} + \mathbf{W}_{jk}\mathbf{S}^*_{(\mathbf{\Omega}_{kt})_{t=0}^{l_k}}, \quad j,k\in\mathbb{N}_{0,m},$$

or, equivalently,

$$\mathbf{P}_\Delta = \mathbf{SW} + \mathbf{WS}^* = 2\mathfrak{Re}(\mathbf{SW}). \tag{7}$$

Moreover, if Ω is a solution of the Problem (MNP) then in view of (6) and (1) we have

$$\mathbf{p}_{st}^{(jk)} = \frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s\partial w^t}\left[\frac{1}{1-vw}(\Omega(v)+[\Omega(\overline{w})]^*)\right]_{\substack{v=\beta_j\\ w=\overline{\beta_k}}}. \tag{8}$$

Keeping this in mind and (2), based on a complex $q\times q$ matrix-valued function Ω which is holomorphic in $\mathbb{D}$ and a sequence $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}$ we also write

$$\mathbf{P}_{m,\beta,l}^{(\Omega)} := (\mathbf{P}_{jk})_{j,k=0}^m \quad \left(\text{respectively, } \widetilde{\mathbf{P}}_{m,\beta,l}^{(\Omega)} := (\widetilde{\mathbf{P}}_{jk})_{j,k=0}^m\right), \tag{9}$$

where $\mathbf{P}_{jk} \in \mathbb{C}^{(l_j+1)q\times(l_k+1)q}$ with entries $\mathbf{p}_{st}^{(jk)} \in \mathbb{C}^{q\times q}$ determined by

$$\mathbf{p}_{st}^{(jk)} := \frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{1-vw}\left(\widehat{\Omega}(v) + [\widehat{\Omega}(\overline{w})]^*\right)\right]_{\substack{v=\beta_j\\ w=\overline{\beta_k}}} \tag{10}$$

and where $\widetilde{\mathbf{P}}_{jk} \in \mathbb{C}^{(l_j+1)q\times(l_k+1)q}$ with entries $\widetilde{\mathbf{p}}_{st}^{(jk)} \in \mathbb{C}^{q\times q}$ determined by

$$\widetilde{\mathbf{p}}_{st}^{(jk)} := \frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{1-vw}\left([\widehat{\Omega}(\overline{v})]^* + \widehat{\Omega}(w)\right)\right]_{\substack{v=\overline{\beta_j}\\ w=\beta_k}}. \tag{11}$$

A well-known criteria for the existence of a function $\Omega \in \mathscr{C}_q(\mathbb{D})$ fulfilling (1) is the following (see, e.g., [5, Corollary 2.7]):

Theorem 0. *For a given data set Δ as in* (4), *the Problem* (*MNP*) *has a solution if and only if the generalized Schwarz–Pick–Potapov block matrix of the first kind $\mathbf{P}_\Delta$* (*respectively*, $\widetilde{\mathbf{P}}_\Delta$) *is nonnegative Hermitian.*

As a conclusion of Theorem 0, we remark here the following characterization of the class $\mathscr{C}_q(\mathbb{D})$:

Theorem 1. *Let Ω be a complex $q \times q$ matrix-valued function defined on $\mathbb{D}$ which is holomorphic in $\mathbb{D}$, let $\widehat{\Omega}$ be given by* (2), *and let m be a nonnegative integer. The following statements are equivalent*:

(i) *The matrix-valued function Ω belongs to the Carathéodory class $\mathscr{C}_q(\mathbb{D})$.*
(ii) *For all $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}$ and all $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ the matrix $\mathbf{P}_{m,\beta,l}^{(\Omega)}$ is nonnegative Hermitian.*
(iii) *For all $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}$ and all $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ the matrix $\widetilde{\mathbf{P}}_{m,\beta,l}^{(\Omega)}$ is nonnegative Hermitian.*

Note that the considerations in Section 4 imply a proof of Theorem 1 without recourse to Theorem 0.

One can see that, in view of (7) and (8) we have

$$\mathbf{P}_{m,\beta,l}^{(\Omega)} = 2\Re\mathrm{e}\mathbf{S}_{l_0}^{(\Omega)} \quad \text{if } m = 0 \text{ and } \beta_0 = 0 \tag{12}$$

for each $l_0 \in \mathbb{N}_0$ and, in addition, for each $m \in \mathbb{N}_0$ and each $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}$ we get the classical Schwarz–Pick–Potapov block matrices of the first kind

$$\mathbf{P}_{m,\beta,l}^{(\Omega)} = \left(\frac{1}{1-\beta_j\overline{\beta_k}}\left(\widehat{\Omega}(\beta_j) + [\widehat{\Omega}(\beta_k)]^*\right)\right)_{j,k=0}^m \quad \text{if } \sum_{k=0}^m l_k = 0 \tag{13}$$

respectively,

$$\widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l} = \left(\frac{1}{1-\overline{\beta_j}\beta_k}\left([\widehat{\Omega}(\beta_j)]^* + \widehat{\Omega}(\beta_k)\right)\right)_{j,k=0}^{m} \quad \text{if } \sum_{k=0}^{m} l_k = 0$$

(see, e.g., [9, Theorem 2.3.1]). The consideration of the extended function $\widehat{\Omega}$ instead of Ω is one of the cornerstones of Potapov's "Method of Fundamental Matrix Inequality". In this way, his school obtained far-reaching generalizations of the classical Schwarz–Pick inequalities (see [20,11,8,17]).

As a main result of this paper, we obtain the following information on the rank of generalized Schwarz–Pick–Potapov block matrices of the first kind. Roughly speaking, this result shows that the rank of such kind of matrices is determined by the number $n+1$ of given data ($q \times q$ matrices) in the Problem (MNP).

Theorem 2. *Let $\Omega \in \mathscr{C}_q(\mathbb{D})$ and let $m \in \mathbb{N}_0$. Then*

$$\operatorname{rank} \mathbf{P}^{(\Omega)}_{m,\beta,l} = \operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_n] \quad \textit{and} \quad \operatorname{rank} \widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l} = \operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_n]$$

for all $(\beta_k)_{k=0}^m \in \mathscr{T}^{\#}_{0,m}$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$, where n is given as in (5).

In view of (13), Theorem 2 is an extension of [16, Theorem 2.2] and, in particular, of the well-known fact that (see, e.g., [9, Proposition 2.1.3]), for each $\Omega \in \mathscr{C}_q(\mathbb{D})$ we have $\operatorname{rank}\,[\Re\mathrm{e}\Omega(w)] = \operatorname{rank}\,[\Re\mathrm{e}\Omega(0)],\ w \in \mathbb{D}$.

Referring to the Problem (MNP), we derive from Theorem 2 and some well-known results on the trigonometric moment problem (see, e.g., [14]) the following statement (where $\mathbf{S}^{(\Omega)}_{-1} := \mathbf{0}_{q\times q}$).

Theorem 3. *Let $m \in \mathbb{N}_0$, let $(\beta_k)_{k=0}^m$ be a sequence of distinct points belonging to $\mathbb{D}$, let $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$, and let n be the number defined by* (5). *Further, let $\Omega \in \mathscr{C}_q(\mathbb{D})$ be a solution of the Problem* (*MNP*). *Then*:

(a) *The following statements are equivalent*:
  (i) *Ω is the unique solution of the Problem* (*MNP*).
  (ii) *The equality* $\operatorname{rank} \mathbf{P}^{(\Omega)}_{m,\beta,l} = \operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_{n-1}]$ *holds.*
  (iii) *The equality* $\operatorname{rank} \widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l} = \operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_{n-1}]$ *holds.*

(b) *If* (i) *is fulfilled then* $\operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_k] = \operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_{n-1}]$ *for each $k \in \mathbb{N}_{n,\infty}$.*

Observe that, because of Theorem 2 the right-hand side of the equalities in Theorem 3 (ii) (respectively, (iii)) coincides with the rank of the upper left $nq \times nq$ block of the matrix $\mathbf{P}^{(\Omega)}_{m,\beta,l}$ (respectively, $\widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l}$). In view of (6) and (8), this block matrix can be computed through the given data in the Problem (MNP).

Moreover, in connection with Theorem 2 and (12), from the statement of uniqueness in Theorem 3 the next one immediately follows. (Here, the complex $q \times q$ matrices $\mathbf{\Omega}_{kt}$, $t \in \mathbb{N}_{0,l_k}$, $k \in \mathbb{N}_{0,m}$, are not fixed but appropriately chosen.)

Corollary 4. *Let $\Omega \in \mathscr{C}_q(\mathbb{D})$, let $\Omega(w) = \sum_{t=0}^{\infty} \mathbf{A}_t w^t$, $w \in \mathbb{D}$, be its Taylor expansion at $v = 0$, and let $n \in \mathbb{N}_0$. The following statements are equivalent*:

(i) *The choice $\mathbf{A}_{n+1}, \mathbf{A}_{n+2}, \ldots$ of complex $q \times q$ matrices is the unique extension of $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ such that the expression $\sum_{t=0}^{\infty} \mathbf{A}_t w^t$, $w \in \mathbb{D}$, is a Taylor expansion at $v = 0$ of a $q \times q$ Carathéodory function.*

(ii) *There exist an $m \in \mathbb{N}_{0,n}$, some numbers $l_0, l_1, \ldots, l_m \in \mathbb{N}_{0,n}$ which fulfill* (5), *and a sequence $(\beta_k)_{k=0}^m$ of distinct points belonging to $\mathbb{D}$ such that Ω is the unique solution of the Problem* (*MNP*).

(iii) *For all $m \in \mathbb{N}_0$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ satisfying* (5), *and all sequences $(\beta_k)_{k=0}^m$ of distinct points belonging to $\mathbb{D}$ the function Ω is the unique solution of the appropriate Problem* (*MNP*).

An application of Theorem 3, (8), and Theorem 0 provides also the following well-known result on uniqueness for the Problem (MNP) in the scalar case $q = 1$ (cf. [5, Theorem 3.7]).

Corollary 5. *For a given data set Δ as in* (4) *with $q = 1$, the Problem* (*MNP*) *has a unique solution if and only if the generalized Schwarz–Pick–Potapov block matrix of the first kind $\mathbf{P}_\Delta$* (*respectively, $\widetilde{\mathbf{P}}_\Delta$*) *is nonnegative Hermitian and singular.*

Note that, in view of Theorem 3, for the matrix case with $q \neq 1$ the condition that the generalized Schwarz–Pick–Potapov block matrix of the first kind $\mathbf{P}_\Delta$ (respectively, $\widetilde{\mathbf{P}}_\Delta$) is nonnegative Hermitian and singular is not sufficient for the uniqueness of the corresponding Problem (MNP).

4. Proof of the main results

In this section, we give now the proof of the main results presented above. In fact, we demonstrate Theorems 2 and 3, and in passing also Theorem 1 (without recourse to Theorem 0).

The proof of Theorem 2 is based on a well-known connection between $q \times q$ Carathéodory functions and $q \times q$ nonnegative Hermitian-valued Borel measures on the unit circle $\mathbb{T}$. Let $\mathfrak{B}_1$ be the σ-algebra of all Borel subsets of $\mathbb{C}$, let $\mathfrak{B}_\mathbb{T} := \mathfrak{B}_1 \cap \mathbb{T}$, and let $\mathscr{M}_{\geq}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ be the set of all $q \times q$ nonnegative Hermitian-valued Borel measures on $\mathbb{T}$, i.e. the set of all countably additive mappings from $\mathfrak{B}_\mathbb{T}$ into the set of nonnegative Hermitian $q \times q$ matrices.

For complex numbers v, w, and z, an easy calculation shows

$$\frac{1}{1-vw}\left(\frac{z+v}{z-v}+\frac{\bar{z}+w}{\bar{z}-w}\right)=\frac{2}{(1-v\bar{z})(1-wz)},\quad \text{if } z\in\mathbb{T},\ vw\neq 1. \tag{14}$$

Applying (14) for $v\in\mathbb{D}$ and $w:=\bar{v}$, it is not hard to accept that, if F belongs to $\mathscr{M}^q_{\geqslant}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ then the mapping $\Omega_F:\mathbb{D}\to\mathbb{C}^{q\times q}$ defined by

$$\Omega_F(v):=\int_{\mathbb{T}}\frac{z+v}{z-v}F(\mathrm{d}z)$$

belongs to $\mathscr{C}_q(\mathbb{D})$ and satisfies $\Im\mathfrak{m}\Omega_F(0)=\mathbf{0}_{q\times q}$. In this case Ω_F is called the *Riesz–Herglotz transform of F*.

Conversely, if $\Omega\in\mathscr{C}_q(\mathbb{D})$ then from a matricial version of a famous theorem due to Riesz and Herglotz we know (see, e.g., [9, Theorem 2.2.2]) that there is a unique measure $F\in\mathscr{M}^q_{\geq}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ such that

$$\Omega(v)-\mathrm{i}\,\Im\mathfrak{m}\Omega(0)=\int_{\mathbb{T}}\frac{z+v}{z-v}F(\mathrm{d}z)$$

for all $v\in\mathbb{D}$. This unique $F\in\mathscr{M}^q_{\geqslant}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ is said to be the *Riesz–Herglotz measure associated with Ω*. In this case, by using (2) even there is

$$\widehat{\Omega}(w)-\mathrm{i}\Im\mathfrak{m}\Omega(0)=\int_{\mathbb{T}}\frac{z+w}{z-w}F(\mathrm{d}z),\quad w\in\mathbb{C}\setminus\mathbb{T}. \tag{15}$$

If F is the Riesz–Herglotz measure associated with a given $q\times q$ Carathéodory function Ω then Ω admits the representation

$$\Omega(w)-\mathrm{i}\,\Im\mathfrak{m}\Omega(0)=\mathbf{\Gamma}_0^{(F)}+2\sum_{j=1}^{\infty}\mathbf{\Gamma}_j^{(F)}w^j,\quad w\in\mathbb{D},$$

where $\mathbf{\Gamma}_k^{(F)}:=\int_{\mathbb{T}}z^{-k}F(\mathrm{d}z),\ k\in\mathbb{Z}$, are the *Fourier coefficients of F* (cf. [9, Section 2.2]). Consequently, we have

$$\Re\mathfrak{e}\mathbf{S}_n^{(\Omega)}=\mathbf{T}_n^{(F)} \tag{16}$$

for each $n\in\mathbb{N}_0$, where $\mathbf{T}_n^{(F)}$ is the nth block Toeplitz matrix associated with the sequence $(\mathbf{\Gamma}_k^{(F)})_{k\in\mathbb{Z}}$ of the Fourier coefficients of F, i.e.

$$\mathbf{T}_n^{(F)}:=\left(\mathbf{\Gamma}_{j-k}^{(F)}\right)_{j,k=0}^{n}. \tag{17}$$

For more basic facts on the integration theory concerning nonnegative Hermitian measures we refer to [21].

The main idea to prove Theorem 2 is to consider $\mathbb{C}^{q\times q}$-modules of rational matrix-valued functions which are associated with the Riesz–Herglotz measure of the given $q\times q$ Carathéodory function. Accordingly, in the following we give some brief informations on this particular class of matrix-valued functions. For each $k\in\mathbb{N}_0$, we define $e_k:\mathbb{C}\to\mathbb{C}$ by

$$e_k(w) := w^k \tag{18}$$

and, if $\eta \in \mathbb{C}$, then $\rho_{\eta,k} : \mathbb{C} \to \mathbb{C}$ by

$$\rho_{\eta,k}(w) := (1 - \overline{\eta} w)^k. \tag{19}$$

The notation $\mathscr{R}_{\alpha,0}$ stands for the set of all constant complex-valued functions defined on $\mathbb{C}_0$ and $\pi_{\alpha,0}$ designates the constant function defined on $\mathbb{C}_0$ with value 1. If $n \in \mathbb{N}$ and if $\alpha_1, \alpha_2, \dots, \alpha_n$ are given complex numbers, then let

$$\pi_{\alpha,n} := \prod_{j=1}^{n} \rho_{\alpha_j,1}, \tag{20}$$

and let $\mathscr{R}_{\alpha,n}$ denote the set of all functions f which are meromorphic in $\mathbb{C}_0$ and which admit a representation $f = \frac{1}{\pi_{\alpha,n}} P$ with a complex polynomial P of degree not greater than n. Observe that if $(\alpha_j)_{j=1}^{\infty}$ is a given sequence of complex numbers, for each $n \in \mathbb{N}_0$, the class $\mathscr{R}_{\alpha,n}^{q \times q}$ (i.e. the set of all $q \times q$ matrices each entry of which belongs to $\mathscr{R}_{\alpha,n}$) can be considered as right (respectively, left) $\mathbb{C}^{q \times q}$-module. Furthermore, one can see that, for each $n \in \mathbb{N}$, the set $\widetilde{\mathscr{R}}_{\alpha,n}^{q \times q}$ of all $Z \in \mathscr{R}_{\alpha,n}^{q \times q}$ which can be represented via $Z = \frac{1}{\pi_{\alpha,n}} Q$, where Q is a complex $q \times q$ matrix polynomial of degree not greater than $n - 1$ is both a right $\mathbb{C}^{q \times q}$-submodule of the right $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$ and a left $\mathbb{C}^{q \times q}$-submodule of the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$.

In [16, Theorems 2.7 and 2.8] the following result is obtained. Here and in the following, for a matrix-valued function X defined on a subset G of $\mathbb{C}_0$ with $\mathbb{T} \subseteq G$ we write $\underline{X}$ to denote the restriction of X onto $\mathbb{T}$.

Theorem 6. *Let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, let $n \in \mathbb{N}_0$, and let $(\alpha_j)_{j=1}^{n+1} \in \mathscr{T}_{1,n+1}$. For each basis $\{Y_0, Y_1, \dots, Y_n\}$ of the right (respectively, left) $\mathbb{C}^{q \times q}$-module $\widetilde{\mathscr{R}}_{\alpha,n+1}^{q \times q}$,*

$$\operatorname{rank} \left(\int_{\mathbb{T}} \underline{Y_j^*} \, \mathrm{d}F \, \underline{Y_k} \right)_{j,k=0}^{n} = \operatorname{rank} \mathbf{T}_n^{(F)}$$

$$\left(\textit{respectively}, \operatorname{rank} \left(\int_{\mathbb{T}} \underline{Y_j} \, \mathrm{d}F \, \underline{Y_k^*} \right)_{j,k=0}^{n} = \operatorname{rank} \mathbf{T}_n^{(F)} \right).$$

In the subsequent considerations, Theorem 6 takes a central position. By using the notation (18) and (19), we introduce now the basis of both the right $\mathbb{C}^{q \times q}$-module $\widetilde{\mathscr{R}}_{\alpha,n+1}^{q \times q}$ and the left $\mathbb{C}^{q \times q}$-module $\widetilde{\mathscr{R}}_{\alpha,n+1}^{q \times q}$ which will us give in connection with Theorem 6 a proof of the main result Theorem 2. We put

$$Y_{kt} := \frac{e_t}{\rho_{k,t+1}} \mathbf{I}_q, \quad t \in \mathbb{N}_{0,l_k}, \ k \in \mathbb{N}_{0,m},$$

and, in view of the Problem (MNP), with $(\beta_k)_{k=0}^m \in \mathscr{T}^{\#}_{0,m}$ we form the sequence $(\alpha_j)_{j=1}^{n+1}$ in which β_k appears according to its multiplicity l_k (use $\sum_{r=0}^{-1} l_r = 0$):

$$\alpha_j := \beta_k \quad \text{if } k+1+\sum_{r=0}^{k-1} l_r \leqslant j \leqslant k+1+\sum_{r=0}^{k} l_r, \quad k \in \mathbb{N}_{0,m}. \tag{21}$$

Correspondingly, we renumber the rational matrix-valued functions Y_{kt} by

$$Y_j := Y_{kt}, \quad j = t+k+\sum_{h=0}^{k-1} l_h, \quad t \in \mathbb{N}_{0,l_k}, \;\; k \in \mathbb{N}_{0,m}. \tag{22}$$

Lemma 7. *Let $m \in \mathbb{N}_0$, let $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$, let n be the number defined by (5), and let $(\beta_k)_{k=0}^m \in \mathscr{T}^{\#}_{0,m}$.*

(a) *Let $(\alpha_j)_{j=1}^{n+1}$ be the sequence of complex numbers given by (21) and let $(Y_j)_{j=0}^n$ be the sequence of matrix-valued functions given by (22). Then $\{Y_0, Y_1, \ldots, Y_n\}$ is both a basis of the right $\mathbb{C}^{q\times q}$-module $\widetilde{\mathscr{R}}^{q\times q}_{\alpha,n+1}$ and a basis of the left $\mathbb{C}^{q\times q}$-module $\widetilde{\mathscr{R}}^{q\times q}_{\alpha,n+1}$.*

(b) *Let $\Omega \in \mathscr{C}_q(\mathbb{D})$ and let F be the Riesz–Herglotz measure associated with Ω. Then*

$$\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{1-vw}\left(\widehat{\Omega}(v)+[\widehat{\Omega}(\overline{w})]^*\right)\right]_{\substack{v=\beta_j\\ w=\overline{\beta_k}}} = 2\int_{\mathbb{T}} \underline{Y_{js}^*}\,\mathrm{d}F\,\underline{Y_{kt}}$$

and

$$\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{1-vw}\left([\widehat{\Omega}(\bar{v})]^*+\widehat{\Omega}(w)\right)\right]_{\substack{v=\overline{\beta_j}\\ w=\beta_k}} = 2\int_{\mathbb{T}} \underline{Y_{js}}\,\mathrm{d}F\,\underline{Y_{kt}^*}$$

for each $s \in \mathbb{N}_{0,l_j}$ and each $t \in \mathbb{N}_{0,l_k}$, $j,k \in \mathbb{N}_{0,m}$.

Proof. (a) By a straightforward calculation, one can check that the set $\{Y_0, Y_1, \ldots, Y_n\}$, i.e. $\left\{\frac{e_t}{\rho_{\beta_k,t+1}}\mathbf{I}_q : t \in \mathbb{N}_{0,l_k},\; k \in \mathbb{N}_{0,m}\right\}$, is a right-hand (respectively, left-hand) $\mathbb{C}^{q\times q}$-linear independent system. From (20) we have $\frac{e_0}{\rho_{\beta_0,1}}\mathbf{I}_q = \frac{1}{\pi_{\alpha,1}}\mathbf{I}_q$. If $n > 0$ then, for $k \in \mathbb{N}_{0,m}$ and $t \in \mathbb{N}_{0,l_k}$, we see

$$\frac{e_t}{\rho_{\beta_k,t+1}}\mathbf{I}_q = \frac{1}{\pi_{\alpha,n+1}}P_{k,t}, \quad \text{where } P_{k,t} := e_t\rho_{\beta_k,l_k-t}\left(\prod_{\substack{j=0\\ j\neq k}}^{m} \rho_{\beta_j,l_j+1}\right)\mathbf{I}_q$$

is a $q \times q$ matrix polynomial of degree not greater than n. Thus, for each $k \in \mathbb{N}_{0,m}$ and $t \in \mathbb{N}_{0,l_k}$, the function $\frac{e_t}{\rho_{\beta_k,t+1}}\mathbf{I}_q$ belongs to $\widetilde{\mathscr{R}}^{q\times q}_{\alpha,n+1}$ and the assertion stated in part (a) follows (cf. [16, Lemma 2.5]).

(b) An application of (14) implies

$$\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s\partial w^t}\left[\frac{1}{1-vw}\left(\frac{z+v}{z-v}+\frac{\overline{z}+w}{\overline{z}-w}\right)\right]_{\substack{v=\beta_j\\ w=\overline{\beta_k}}}$$
$$=\frac{2\overline{z}^s z^t}{(1-\beta_j\overline{z})^{s+1}(1-\overline{\beta_k}z)^{t+1}}=2\overline{\left(\frac{e_s(z)}{\rho_{\beta_j,s+1}(z)}\right)}\left(\frac{e_t(z)}{\rho_{\beta_k,t+1}(z)}\right) \tag{23}$$

for $z\in\mathbb{T}$, $s\in\mathbb{N}_{0,l_j}$, $t\in\mathbb{N}_{0,l_k}$, and $j,k\in\mathbb{N}_{0,m}$. Hence, referring to (15) we can conclude the statement of part (b). □

Observe that part (b) of Lemma 7 yields a proof of Theorem 1 (indeed the implications "(i)⇒(ii)" and "(i)⇒(iii)", in which "(ii)⇒(i)" and "(iii)⇒(i)" are evident).

Now we prove Theorems 2 and 3.

Proof of Theorem 2. We consider an arbitrary sequence $(\beta_k)_{k=0}^m\in\mathscr{T}_{0,m}^{\#}$ and arbitrary $l_0,l_1,\dots,l_m\in\mathbb{N}_0$. Furthermore, let n be the number defined as in (5), let $(\alpha_j)_{j=1}^{n+1}$ be given by (21), and let $(Y_j)_{j=0}^n$ be given by (22). From part (a) of Lemma 7 we know that $\{Y_0,Y_1,\dots,Y_n\}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\widetilde{\mathscr{R}}_{\alpha,n+1}^{q\times q}$ and a basis of the left $\mathbb{C}^{q\times q}$-module $\widetilde{\mathscr{R}}_{\alpha,n+1}^{q\times q}$. Thus, an application of Theorem 6, (22), and part (b) of Lemma 7 gives

$$\operatorname{rank}\mathbf{P}_{m,\beta,l}^{(\Omega)}=\operatorname{rank}\mathbf{T}_n^{(F)},$$

where $\mathbf{P}_{m,\beta,l}^{(\Omega)}$ is defined as in (9) and F denotes the Riesz–Herglotz measure associated with Ω. Hence, we get

$$\operatorname{rank}\mathbf{P}_{m,\beta,l}^{(\Omega)}=\operatorname{rank}[\Re\mathrm{e}\,\mathbf{S}_n^{(\Omega)}]$$

from (16). Similarly, in view of (9), Theorem 6, part (b) of Lemma 7, and (16) we can conclude $\operatorname{rank}\widetilde{\mathbf{P}}_{m,\beta,l}^{(\Omega)}=\operatorname{rank}[\Re\mathrm{e}\,\mathbf{S}_n^{(\Omega)}]$ and the proof is complete. □

Proof of Theorem 3. From Theorem 2 we see that (ii) and (iii) are equivalent. Now, we prove the equivalence of (i) and (ii) as well as part (b). In view of the setting $\mathbf{S}_{-1}^{(\Omega)}:=\mathbf{0}_{q\times q}$ and elementary properties of matrix-valued Carathéodory functions (use, e.g., [9, Corollary 2.3.1]) the case $n=0$ is obvious. Now, let $n\in\mathbb{N}$. Further, let $(\alpha_j)_{j=1}^{n+1}$ be the sequence of complex numbers given by (21) and let F be the Riesz–Herglotz measure associated with Ω.

First, we assume that (i) is satisfied. In view of the connection between the sets $\mathscr{C}_q(\mathbb{D})$ and $\mathscr{M}_{\geq}^q(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$, from (i) and Lemma 7 one can see that there is a unique $E\in\mathscr{M}_{\geq}^q(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$ fulfilling the relations

$$\int_{\mathbb{T}}\underline{X^*}\,\mathrm{d}E\underline{Y}=\int_{\mathbb{T}}\underline{X^*}\,\mathrm{d}F\underline{Y},\qquad X,Y\in\widetilde{\mathscr{R}}_{\alpha,n+1}^{q\times q}, \tag{24}$$

namely $E = F$. If, for an arbitrary $H \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, we define the measure $H^{(\alpha,n+1)} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ by

$$H^{(\alpha,n+1)}(A) := \int_A \left(\frac{1}{\underline{\pi_{\alpha,n+1}}}\mathbf{I}_q\right)^* \mathrm{d}H \left(\frac{1}{\underline{\pi_{\alpha,n+1}}}\mathbf{I}_q\right)$$

then $H^{(\alpha,n+1)}$ belongs also to $\mathscr{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ (cf. [15, Lemma 3.1]). By using this, the fact that $\{e_0\mathbf{I}_q, e_1\mathbf{I}_q, \ldots, e_n\mathbf{I}_q\}$ is a basis of the right $\mathbb{C}^{q\times q}$-module of all complex $q \times q$ matrix polynomials of degree not greater than n, and (17) then we obtain that there is a unique $E \in \mathscr{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ fulfilling the equality

$$\mathbf{T}_n^{(E)} = \mathbf{T}_n^{(F^{(\alpha,n+1)})}, \tag{25}$$

namely $E = F^{(\alpha,n+1)}$. By virtue of the interrelation between nonnegative Hermitian $q \times q$ block Toeplitz matrices and measures belonging to $\mathscr{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ (see, e.g., [14, Theorems 4 and 5]), [14, Remark 2] yields the inequality

$$\operatorname{rank} \mathbf{T}_k^{(F^{(\alpha,n+1)})} \leqslant \operatorname{rank} \mathbf{T}_{k+1}^{(F^{(\alpha,n+1)})}, \quad k \in \mathbb{N}_0,$$

and, hence, (25) in connection with some results on the extension of nonnegative Hermitian $q \times q$ block Toeplitz matrices (see, e.g., [14, Theorem 1 and Lemma 6]) imply

$$\operatorname{rank} \mathbf{T}_{k-1}^{(F^{(\alpha,n+1)})} = \operatorname{rank} \mathbf{T}_k^{(F^{(\alpha,n+1)})}, \quad k \in \mathbb{N}_{n,\infty}. \tag{26}$$

Therefore, an application of (16) and [15, Remarks 3.9 and 4.7]) yields

$$\begin{aligned}\operatorname{rank}[\Re\mathrm{e}\mathbf{S}_k^{(\Omega)}] &= \operatorname{rank} \mathbf{T}_k^{(F)} = \operatorname{rank} \mathbf{T}_k^{(F^{(\alpha,n+1)})} \\ &= \operatorname{rank} \mathbf{T}_{n-1}^{(F^{(\alpha,n+1)})} = \operatorname{rank} \mathbf{T}_{n-1}^{(F)} = \operatorname{rank}[\Re\mathrm{e}\mathbf{S}_{n-1}^{(\Omega)}], \quad k \in \mathbb{N}_{n,\infty}.\end{aligned}$$

Consequently, we have obtained the assertion of part (b) and, in view of Theorem 2, also that (ii) is satisfied.

The same arguments in reverse order show that (ii) implies (i). In fact, by using Theorem 2, (16), and [15, Remarks 3.9 and 4.7]) from (ii) we get

$$\operatorname{rank} \mathbf{T}_n^{(F^{(\alpha,n+1)})} = \operatorname{rank} \mathbf{T}_{n-1}^{(F^{(\alpha,n+1)})}$$

and, thus, from [14, Lemma 6] one can conclude (26). Then an application of [14, Theorems 1, 4, and 5] provides that there is a unique $E \in \mathscr{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ fulfilling the equality (25), namely $E = F^{(\alpha,n+1)}$. As already indicated above, this is tantamount to there is a unique $E \in \mathscr{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ fulfilling (24), namely $E = F$. Finally, Lemma 7 shows that (i) is fulfilled. □

5. Generalized Schwarz–Pick–Potapov matrices of the second kind

Now we turn the attention to the generalized Schwarz–Pick–Potapov block matrices of the second kind for matricial Carathéodory functions. In fact, we give a generalization of a theorem which characterizes the class $\mathscr{C}_q(\mathbb{D})$ and which goes essentially back to Kovalishina [17] (see also [9, Theorem 2.3.2]). In addition, for this class of matrices we will also verify some results on rank invariance.

Let $m_1 \in \mathbb{N}_0$, $m_2 \in \mathbb{N}$, $l_0, l_1, \ldots, l_{m_1+m_2} \in \mathbb{N}_0$, and let $\beta_0, \beta_1, \ldots, \beta_{m_1+m_2} \in \mathbb{C}$ be such that the conditions

$$(\beta_j)_{j=0}^{m_1} \in \mathscr{T}_{0,m_1}, \ (\beta_k)_{k=m_1+1}^{m_1+m_2} \in \mathscr{T}_{m_1,m_1+m_2}, \tag{27}$$

and

$$\beta_j \neq \beta_k, \quad j \in \mathbb{N}_{0,m_1}, \ k \in \mathbb{N}_{m_1+1,m_1+m_2} \tag{28}$$

are satisfied. Then for a complex $q \times q$ matrix-valued function Ω defined on $\mathbb{D}$ which is holomorphic in $\mathbb{D}$ we denote

$$\mathbf{Q}^{(\Omega)}_{m_1,m_2,\beta,l} := (\mathbf{Q}_{jk})_{\substack{j=0,1,\ldots,m_1 \\ k=m_1+1,m_1+2,\ldots,m_1+m_2}},$$

where $\mathbf{Q}_{jk} \in \mathbb{C}^{(l_j+1)q\times(l_k+1)q}$ with entries $\mathbf{q}^{(jk)}_{st} \in \mathbb{C}^{q\times q}$ determined by

$$\mathbf{q}^{(jk)}_{st} := \frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{v-w}\left(\widehat{\Omega}(v) - \widehat{\Omega}(w)\right)\right]_{\substack{v=\beta_j \\ w=\beta_k}},$$

and we introduce the *generalized Schwarz–Pick–Potapov block matrix of the second kind* by

$$\mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l} := \begin{pmatrix} \mathbf{P}^{(\Omega)}_{m_1,\beta^{(1)},l^{(1)}} & \mathbf{Q}^{(\Omega)}_{m_1,m_2,\beta,l} \\ \left[\mathbf{Q}^{(\Omega)}_{m_1,m_2,\beta,l}\right]^* & \widetilde{\mathbf{P}}^{(\Omega)}_{m_2-1,\beta^{(2)},l^{(2)}} \end{pmatrix},$$

where $\beta^{(1)}$ and $l^{(1)}$ (respectively, $\beta^{(2)}$ and $l^{(2)}$) stand for the sequences $(\beta_k)_{k=0}^{m_1}$ and $(l_k)_{k=0}^{m_1}$ (respectively, $(\beta_{m_1+1+k})_{k=0}^{m_2-1}$ and $(l_{m_1+1+k})_{k=0}^{m_2-1}$) as well as $\mathbf{P}^{(\Omega)}_{m_1,\beta^{(1)},l^{(1)}}$ (respectively, $\widetilde{\mathbf{P}}^{(\Omega)}_{m_2-1,\beta^{(2)},l^{(2)}}$) is defined as in (9) and (10) (respectively, (9) and (11)).

Lemma 8. *Let $\Omega \in \mathscr{C}_q(\mathbb{D})$ and let F be the Riesz–Herglotz measure associated with Ω. Further, let $s, t \in \mathbb{N}_0$. Then*

$$\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{v-w}\left(\widehat{\Omega}(v) - \widehat{\Omega}(w)\right)\right]_{\substack{v=\beta_0 \\ w=\beta_1}}$$

$$= \frac{2s!t!}{(-\beta_1)^{t+1}}\int_{\mathbb{T}}\left(\frac{e_s}{\underline{\rho_{\beta_0,s+1}}}\mathbf{I}_q\right)^* \mathrm{d}F\left(\frac{e_0}{\underline{\rho_{\frac{1}{\overline{\beta_1}},t+1}}}\mathbf{I}_q\right)$$

for each $\beta_0 \in \mathbb{C} \setminus \mathbb{T}$ and each $\beta_1 \in \mathbb{C} \setminus (\mathbb{T} \cup \beta_0 \cup \{0\})$ as well as

$$\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{1-vw}\left([\widehat{\Omega}(\bar{v})]^* + \widehat{\Omega}(w)\right)\right]_{\substack{v=\overline{\beta_1}\\ w=\beta_2}}$$

$$= \frac{2s!t!}{(-\overline{\beta_1})^{s+1}(-\beta_2)^{t+1}} \int_{\mathbb{T}} \left(\frac{e_0}{\underline{\rho_{\frac{1}{\overline{\beta_1}},s+1}}}\mathbf{I}_q\right)^* \mathrm{d}F \left(\frac{e_0}{\underline{\rho_{\frac{1}{\overline{\beta_2}},t+1}}}\mathbf{I}_q\right)$$

for all $\beta_1, \beta_2 \in \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$ *satisfying* $1 \neq \overline{\beta_1}\beta_2$.

Proof. An easy calculation shows

$$\frac{1}{v-w}\left(\frac{z+v}{z-v} - \frac{z+w}{z-w}\right) = \frac{2z}{(z-v)(z-w)}$$

for distinct points z, v, w belonging to $\mathbb{C}$. Therefore, in view of (18) and (19) we obtain

$$\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{v-w}\left(\frac{z+v}{z-v} - \frac{z+w}{z-w}\right)\right]_{\substack{v=\beta_0\\ w=\beta_1}}$$

$$= \frac{2z}{(z-\beta_0)^{s+1}(z-\beta_1)^{t+1}} = \frac{2\overline{z}^s}{(1-\beta_0\overline{z})^{s+1}(1-\frac{1}{\beta_1}z)^{t+1}(-\beta_1)^{t+1}}$$

$$= \frac{2}{(-\beta_1)^{t+1}} \overline{\left(\frac{e_s(z)}{\rho_{\beta_0,s+1}(z)}\right)} \left(\frac{e_0(z)}{\rho_{\frac{1}{\overline{\beta_1}},t+1}(z)}\right) \tag{29}$$

for $z \in \mathbb{T}$, $\beta_0 \in \mathbb{C} \setminus \mathbb{T}$, and $\beta_1 \in \mathbb{C} \setminus (\mathbb{T} \cup \beta_0 \cup \{0\})$. Moreover, from (23) we see

$$\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\left[\frac{1}{1-vw}\left(\frac{\overline{z}+v}{\overline{z}-v} + \frac{z+w}{z-w}\right)\right]_{\substack{v=\overline{\beta_1}\\ w=\beta_2}}$$

$$= \frac{2z^s\overline{z}^t}{(1-\overline{\beta_1}z)^{s+1}(1-\beta_2\overline{z})^{t+1}}$$

$$= \frac{2}{(-\overline{\beta_1})^{s+1}(1-\frac{1}{\overline{\beta_1}}\overline{z})^{s+1}(1-\frac{1}{\overline{\beta_2}}z)^{t+1}(-\beta_2)^{t+1}}$$

$$= \frac{2}{(-\overline{\beta_1})^{s+1}(-\beta_2)^{t+1}} \overline{\left(\frac{e_0(z)}{\rho_{\frac{1}{\overline{\beta_1}},s+1}(z)}\right)} \left(\frac{e_0(z)}{\rho_{\frac{1}{\overline{\beta_2}},t+1}(z)}\right) \tag{30}$$

for $z \in \mathbb{T}$ and $\beta_1, \beta_2 \in \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$ which satisfy $1 \neq \overline{\beta_1}\beta_2$. Hence, referring to (29), (30), and (15) we can conclude the assertions. □

Theorem 9. *Let Ω be a complex $q \times q$ matrix-valued function defined on $\mathbb{D}$ which is holomorphic in $\mathbb{D}$ and let $\widehat{\Omega}$ be given by (2). Further, let $m_1 \in \mathbb{N}_0$, let $m_2 \in \mathbb{N}$, and let $l_0, l_1, \ldots, l_{m_1+m_2} \in \mathbb{N}_0$. Then:*

(a) *The following statements are equivalent:*
  (i) *Ω belongs to the Carathéodory class $\mathscr{C}_q(\mathbb{D})$.*
  (ii) *For every choice of $\beta_0, \beta_1, \ldots, \beta_{m_1+m_2} \in \mathbb{C}$ which satisfies (27) and (28), the matrix $\mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l}$ is nonnegative Hermitian.*

(b) *If $\Omega \in \mathscr{C}_q(\mathbb{D})$ and if $(\beta_k)_{k=0}^{m_1+m_2} \in \mathscr{T}^{\#}_{0,m_1+m_2}$ then*

$$\operatorname{rank} \mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l} = \operatorname{rank}\,[\Re\mathrm{e}\mathbf{S}^{(\Omega)}_{\tilde{n}}],$$

where $\tilde{n} := m_1 + m_2 + \sum_{k=0}^{m_1+m_2} l_k$.

Proof. The implication "(ii)⇒(i)" is obvious. Now, we suppose $\Omega \in \mathscr{C}_q(\mathbb{D})$. Furthermore, let $\beta_0, \beta_1, \ldots, \beta_{m_1+m_2} \in \mathbb{C}$ which satisfy (27) and (28). First we consider the case that $\beta_k \neq 0$ for each, $k \in \mathbb{N}_{m_1+1,m_1+m_2}$. Therefore, the block diagonal matrix $\mathbf{X} := \operatorname{diag}(\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_{\tilde{n}})$ where (use $\sum_{r=0}^{-1} l_r = 0$)

$$\mathbf{X}_{t+k+\sum_{r=0}^{k-1} l_r} := \begin{cases} \mathbf{I}_q & \text{if } t \in \mathbb{N}_{0,l_k},\ k \in \mathbb{N}_{0,m_1}, \\ \frac{1}{(-\beta_k)^{t+1}}\mathbf{I}_q & \text{if } t \in \mathbb{N}_{0,l_k},\ k \in \mathbb{N}_{m_1+1,m_1+m_2} \end{cases}$$

is nonsingular, the sequence $(\tau_j)_{j=1}^{m_1+m_2+1}$ given by

$$\tau_{j+1} := \begin{cases} \beta_j & \text{if } j \in \mathbb{N}_{0,m_1}, \\ \frac{1}{\overline{\beta_j}} & \text{if } j \in \mathbb{N}_{m_1+1,m_1+m_2} \end{cases}$$

belongs to $\mathscr{T}_{1,m_1+m_2+1}$, and, in view of Lemmas 7 and 8, the matrix $\mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l}$ admits the representation

$$\mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l} = 2\mathbf{X}^* \left(\int_{\mathbb{T}} \underline{Z_j^*}\, \mathrm{d}F\, \underline{Z_k} \right)_{j,k=0}^{\tilde{n}} \mathbf{X},$$

where the sequence $(Z_j)_{j=0}^{\tilde{n}}$ is given by

$$Z_{t+k+\sum_{r=0}^{k-1} l_r} := \begin{cases} \frac{e_t}{\rho_{\tau_k,t+1}}\mathbf{I}_q & \text{if } t \in \mathbb{N}_{0,l_k},\ k \in \mathbb{N}_{0,m_1}, \\ \frac{e_0}{\rho_{\tau_k,t+1}}\mathbf{I}_q & \text{if } t \in \mathbb{N}_{0,l_k},\ k \in \mathbb{N}_{m_1+1,m_1+m_2}. \end{cases}$$

Hence, the matrix $\mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l}$ is nonnegative Hermitian. Moreover, in the special case $(\beta_k)_{k=0}^{m_1+m_2} \in \mathscr{T}^{\#}_{0,m_1+m_2}$ with $\beta_k \neq 0$, $k \in \mathbb{N}_{m_1+1,m_1+m_2}$ by a straightforward calculation one can check that $\{Z_0, Z_1, \ldots, Z_{\tilde{n}}\}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\widetilde{\mathscr{R}}^{q\times q}_{\alpha,\tilde{n}+1}$, where we form here the sequence $(\alpha_j)_{j=1}^{\tilde{n}+1}$ in which τ_k appears according to its multiplicity l_k:

$$\alpha_j := \tau_k \quad \text{if } k+1+\sum_{r=0}^{k-1} l_r \leqslant j \leqslant k+1+\sum_{r=0}^{k} l_r, \quad k \in \mathbb{N}_{0,m_1+m_2}.$$

Consequently, from Theorem 6 and (16) we obtain the identity

$$\operatorname{rank} \mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l} = \operatorname{rank} [\Re\mathrm{e}\mathbf{S}^{(\Omega)}_{\tilde{n}}].$$

Thus, in the case $\beta_k \neq 0$, $k \in \mathbb{N}_{m_1+1,m_1+m_2}$ the statements are verified. If $\beta_k = 0$ for a $k \in \mathbb{N}_{m_1+1,m_1+m_2}$ then (28) implies $\beta_j \neq 0$ for all $j \in \mathbb{N}_{0,m_1}$ such that the assertion can be similarly derived from Lemmas 7, 8, Theorem 6 and (16). We omit the details (see, e.g., [16, Remark 2.14] regarding the special case $l_k = 0$, $k \in \mathbb{N}_{0,m_1+m_2}$). □

Note that, similar to Theorem 2 (see Theorem 3), part (b) of Theorem 9 immediately gives a characterization of the case of exactly one solution for the finite multiple Nevanlinna–Pick interpolation problem.

6. Some conclusions out of the main results

In this section we discuss some special cases of the main results. In the course of this, we restrict the considerations regarding the generalized Schwarz–Pick–Potapov block matrix $\mathbf{P}^{(\Omega)}_{m,\beta,l}$. Similarly, one can obtain statements referring to $\widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l}$ and $\mathbf{R}^{(\Omega)}_{m_1,m_2,\beta,l}$, respectively.

Special attention is concentrated on the so-called nondegenerate $q \times q$ Carathéodory functions (cf. [9, Section 3.4]). If $n \in \mathbb{N}_0$, then a function $\Omega \in \mathscr{C}_q(\mathbb{D})$ is said to be *nondegenerate of order n* (respectively, *degenerate of order n*) if the matrix $\Re\mathrm{e}\mathbf{S}^{(\Omega)}_n$ is nonsingular (respectively, singular). The main results above provide a characterization of nondegenerate $q \times q$ Carathéodory functions as follows:

Proposition 10. *Let $\Omega \in \mathscr{C}_q(\mathbb{D})$ and let $n \in \mathbb{N}_0$. The following statements are equivalent*:

(i) *The $q \times q$ Carathéodory function Ω is nondegenerate of order n.*
(ii) *There are an $m \in \mathbb{N}_{0,n}$, numbers $l_0, l_1, \ldots, l_m \in \mathbb{N}_{0,n}$ which fulfill* (5), *and a sequence $(\beta_k)_{k=0}^m \in \mathscr{T}^{\#}_{0,m}$ such that the matrix $\mathbf{P}^{(\Omega)}_{m,\beta,l}$ is nonsingular.*
(iii) *There are an $m \in \mathbb{N}_{0,n}$, numbers $l_0, l_1, \ldots, l_m \in \mathbb{N}_{0,n}$ which fulfill* (5), *and a sequence $(\beta_k)_{k=0}^m \in \mathscr{T}^{\#}_{0,m}$ such that the matrix $\widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l}$ is nonsingular.*
(iv) *For each $m \in \mathbb{N}_0$ and each choice of $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ satisfying* (5), *and each sequence $(\beta_k)_{k=0}^m \in \mathscr{T}^{\#}_{0,m}$ the matrices $\mathbf{P}^{(\Omega)}_{m,\beta,l}$ and $\widetilde{\mathbf{P}}^{(\Omega)}_{m,\beta,l}$ are both positive Hermitian.*

Proof. Apply Theorems 1 and 2. □

Keeping Theorem 0 and (8) in mind, in view of the Problem (MNP), Proposition 10 shows for instance the following.

Corollary 11. *Let Δ be a given data as in (4) and let n be the number defined by (5). The following statements are equivalent*:

(i) *The matrix $\mathbf{P}_\Delta$ is positive Hermitian.*
(ii) *There is a solution of the Problem (MNP) which is nondegenerate of order n.*
(iii) *Every solution of the Problem (MNP) is nondegenerate of order n.*

Take into consideration the constant function $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ with value $\frac{1}{2}\mathbf{A}$, where $\mathbf{A}$ is positive Hermitian (following the way as in [16, Corollaries 2.12 and 2.13]), from Proposition 10 one can obtain the two subsequent presented results. The first is an example for a generalized Schwarz–Pick–Potapov block matrix which is positive Hermitian and the second gives a sufficient condition for the nondegeneracy of a $q \times q$ Carathéodory function.

Example 12. Let $m \in \mathbb{N}_0$, let $(\beta_k)_{k=0}^m$ be a sequence of distinct points belonging to $\mathbb{D}$, and let $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$. If $\mathbf{A}$ is a positive Hermitian complex $q \times q$ matrix then the matrix $\mathbf{P}_{m,\beta,l} := (\mathbf{P}_{jk})_{j,k=0}^m$, where $\mathbf{P}_{jk} \in \mathbb{C}^{(l_j+1)q\times(l_k+1)q}$ with entries $\mathbf{p}_{st}^{(jk)} \in \mathbb{C}^{q\times q}$ determined by

$$\mathbf{p}_{st}^{(jk)} := \sum_{r=0}^{\min\{s,t\}} \frac{(s+t-r)!}{(s-r)!r!(t-r)!} \frac{\beta_j^{t-r}\overline{\beta_k^{s-r}}}{(1-\beta_j\overline{\beta_k})^{s+t+1-r}}\mathbf{A},$$

is positive Hermitian as well.

Corollary 13. *Let $m \in \mathbb{N}_0$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ be such that the number n defined by (5) belongs to $\mathbb{N}$. Moreover, let $\Omega \in \mathscr{C}_q(\mathbb{D})$ be such that the following two conditions are satisfied*:

(I) *There is a $v \in \mathbb{C}_0 \setminus \mathbb{T}$ such that the matrix $\Re\mathrm{e}\widehat{\Omega}(v)$ is nonsingular.*
(II) *There exists a sequence $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$ such that $\widehat{\Omega}(\beta_k) = \widehat{\Omega}(\beta_0)$ for each $k \in \mathbb{N}_{1,m}$ and $\widehat{\Omega}^{(t)}(\beta_k) = \mathbf{0}_{q\times q}$ for each $t \in \mathbb{N}_{1,l_k}$, $k \in \mathbb{N}_{0,m}$.*

Then the $q \times q$ Carathéodory function Ω is nondegenerate of order n.

If Ω is a $q \times q$ Carathéodory function which has a nonsingular value then it is well known that the inverse Ω^{-1} of this function also belongs to $\mathscr{C}_q(\mathbb{D})$ and $\operatorname{rank}[\Re\mathrm{e}\mathbf{S}_n^{(\Omega^{-1})}] = \operatorname{rank}[\Re\mathrm{e}\mathbf{S}_n^{(\Omega)}]$ (see, e.g., [9, Lemmas 2.1.10, 1.1.21, and 1.1.15]). Therefore, from Theorem 2 we get the following:

Corollary 14. *Let $\Omega \in \mathscr{C}_q(\mathbb{D})$ be such that the matrix $\Omega(w_0)$ is nonsingular for a $w_0 \in \mathbb{D}$. Then the matrix $\Omega(w)$ is nonsingular for all $w \in \mathbb{D}$, the matrix-valued function Ω^{-1} belongs to $\mathscr{C}_q(\mathbb{D})$ as well, and*

$$\operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\Omega^{-1})} = \operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\Omega)}$$

for every choice of $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$ *and* $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$.

Now we will turn the attention to the matrix-valued Carathéodory functions the Riesz–Herglotz measure of which is atomic (compare [15, Section 6] and [16, Section 3]). Henceforth, we consider a function $\Psi : \mathbb{D} \to \mathbb{C}^{q\times q}$ which is defined by

$$\Psi(w) := \sum_{u=1}^{r} \frac{z_u + w}{z_u - w} \mathbf{A}_u + \mathrm{i}\mathbf{H}, \tag{31}$$

where $r \in \mathbb{N}$, $(\mathbf{A}_u)_{u=1}^r$ is a sequence of nonnegative Hermitian complex $q\times q$ matrices, $(z_u)_{u=1}^r$ is a sequence of unimodular complex numbers, and $\mathbf{H}$ is a Hermitian complex $q \times q$ matrix (and which is obvious a Carathéodory function). Moreover, if $m \in \mathbb{N}_0$, if $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$, and if $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ then, in view of (9), (10) and (23), we have a matrix $\mathbf{P}_{m,\beta,l}^{(\Psi)} = (\mathbf{P}_{jk})_{j,k=0}^m$, where $\mathbf{P}_{jk} \in \mathbb{C}^{(l_j+1)q\times(l_k+1)q}$ with entries $\mathbf{p}_{st}^{(jk)} \in \mathbb{C}^{q\times q}$ determined by

$$\mathbf{p}_{st}^{(jk)} = \sum_{u=1}^{r} \frac{2z_u^{t-s}}{(1-\beta_j\overline{z_u})^{s+1}(1-\overline{\beta_k}z_u)^{t+1}} \mathbf{A}_u. \tag{32}$$

Clearly, the results stated in Theorems 2 and 9 can be specified for the generalized Schwarz–Pick–Potapov block matrices which are constructed made of Ψ. Besides one can give a more detailed estimation of the rank for the special case of matricial Carathéodory functions associated with atomic nonnegative Hermitian-valued Borel measures. Indeed, in view of (31), (13), Theorem 2 and a similar result referring to Schwarz–Pick–Potapov block matrices (see [16, Remark 3.1]), one can obtain the following statement. In particular, it comprises an example for a generalized Schwarz–Pick–Potapov block matrix with the highest degree of degeneracy.

Example 15. Let $r \in \mathbb{N}$, let $(\mathbf{A}_u)_{u=1}^r$ be a sequence of nonnegative Hermitian complex $q \times q$ matrices, let $z_1, z_2, \ldots, z_r \in \mathbb{T}$, let $\mathbf{H}$ be a Hermitian complex $q \times q$ matrix, and let $\Psi : \mathbb{D} \to \mathbb{C}^{q\times q}$ be defined by (31). For, each sequence $m \in \mathbb{N}_0$, each $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$, and each choice of $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$, we have

$$\max_{u\in\{1,\ldots,r\}} [\operatorname{rank} \mathbf{A}_u] \leqslant \operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\Psi)} \leqslant \sum_{u=1}^{r} \operatorname{rank} \mathbf{A}_u$$

and, in particular, if $n \geq r-1$, we obtain

$$\operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\Psi)} = \operatorname{rank} [\Re\mathrm{e}\mathbf{S}_{r-1}^{(\Psi)}] \tag{33}$$

as well as, in the special case, if each of the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r$ is nonsingular and if the points $z_1, z_2, \ldots, z_r$ are distinct then

$$\operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\Psi)} = \begin{cases} (n+1)q & \text{if } 0 \leqslant n \leq r-1, \\ rq & \text{if } n \geqslant r, \end{cases}$$

where n is the number given in (5).

If we consider a finite multiple Nevanlinna–Pick interpolation problem with respect to functions of the form (31) then the results above yield the following statement of uniqueness.

Proposition 16. *Let $r \in \mathbb{N}$, let $(z_u)_{u=1}^r$ be a sequence of distinct points belonging to the unit circle $\mathbb{T}$, let $(\mathbf{A}_u)_{u=1}^r$ be a sequence of nonnegative Hermitian complex $q \times q$ matrices, and let $\mathbf{H}$ be a Hermitian complex $q \times q$ matrix. Further, let $m \in \mathbb{N}_0$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ be such that $n = r$, where n is the number defined by (5), let $(\beta_k)_{k=0}^m$ be a sequence of distinct points belonging to the open unit disk $\mathbb{D}$, let*

$$\mathbf{\Omega}_{k0} := \sum_{u=1}^r \frac{z_u + \beta_k}{z_u - \beta_k} \mathbf{A}_u + \mathrm{i}\mathbf{H}, \quad k \in \mathbb{N}_{0,m},$$

and let

$$\mathbf{\Omega}_{kt} := \sum_{u=1}^r \frac{2z_u}{(z_u - \beta_k)^{t+1}} \mathbf{A}_u, \quad t \in \mathbb{N}_{1,l_k}, k \in \mathbb{N}_{0,m}.$$

Then there is a unique $\Omega \in \mathscr{C}_q(\mathbb{D})$ such that (1) is satisfied, namely $\Omega = \Psi$, where Ψ is given by (31).

Proof. Obviously, the function Ψ is a solution of the problem. Furthermore, an application of part (a) of Theorem 3 and (33) shows that Ψ is the unique solution. $\square$

Proposition 16 yields the following characterization of $q \times q$ Carathéodory functions with atomic Riesz–Herglotz measure.

Corollary 17. *Let $\Psi \in \mathscr{C}_q(\mathbb{D})$, let $r \in \mathbb{N}$, let $(z_u)_{u=1}^r$ be a sequence of distinct unimodular complex numbers, and let $(\mathbf{A}_u)_{u=1}^r$ be a sequence of nonnegative Hermitian complex $q \times q$ matrices. Further, let $m \in \mathbb{N}_0$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ be such that $n = r$, where n is the number defined by (5), and let $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$.*

(a) *The following statements are equivalent:*
   (i) *There is a Hermitian complex $q \times q$ matrix $\mathbf{H}$ such that Ψ admits the representation (31) for each $w \in \mathbb{D}$.*
   (ii) *The matrix $\mathbf{P}_{m,\beta,l}^{(\Psi)}$ given by (9) and (10) can be represented via (32).*

(b) *Let* (ii) *be fulfilled. Then there is a unique $q \times q$ Carathéodory function Ω which satisfies the conditions $\mathbf{P}_{m,\beta,l}^{(\Omega)} = \mathbf{P}_{m,\beta,l}^{(\Psi)}$ and $\Im\mathfrak{m}\Omega(0) = \mathbf{0}_{q\times q}$, namely the function $\Omega = \Psi - \mathrm{i}\, \Im\mathfrak{m}\Psi(0)$.*

Let us finish this paper with some corollaries on complex-valued Carathéodory functions which follow right away from Corollary 17, Example 15 and the well-known fact, that φ is a 1×1 Carathéodory function degenerate of order r if and only if the equality

$$\varphi(w) = \sum_{u=1}^{r} \frac{\zeta_u + w}{\zeta_u - w} a_u + \mathrm{i}h, \quad w \in \mathbb{D}, \tag{34}$$

holds for some $\zeta_1, \zeta_2, \ldots, \zeta_r \in \mathbb{T}$, a sequence $(a_u)_{u=1}^r$ of nonnegative real numbers, and a real number h. Observe that in the matrix case with $q \neq 1$, a $q \times q$ Carathéodory function Ψ which is degenerate of order r have not to fulfill, in general, a representation of kind (31). Therefore, the following two results cannot be easy carry forwarded to that case.

Corollary 18. *Let $\varphi \in \mathscr{C}_1(\mathbb{D})$ and let $r \in \mathbb{N}$. Furthermore, let the numbers $m \in \mathbb{N}_0$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ be such that $n = r$, where n is the number defined by* (5), *and let $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$. The following statements are equivalent*:

(i) *φ is degenerate of order r.*
(ii) *There are a sequence $(\zeta_u)_{u=1}^r$ of unimodular complex numbers, a sequence $(a_u)_{u=1}^r$ of nonnegative real numbers and a real number h such that the equality* (34) *holds.*
(iii) $\operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\varphi)} = \operatorname{rank} [\mathfrak{Re}\mathbf{S}_{r-1}^{(\varphi)}]$.
(iv) $\det \mathbf{P}_{m,\beta,l}^{(\varphi)} = 0$.
(v) *There are a sequence $(\zeta_u)_{u=1}^r$ of unimodular complex numbers and a sequence $(a_u)_{u=1}^r$ of nonnegative real numbers such that $\mathbf{P}_{m,\beta,l}^{(\varphi)} = (\mathbf{P}_{jk})_{j,k=0}^m$, where $\mathbf{P}_{jk} \in \mathbb{C}^{(l_j+1)\times(l_k+1)}$ with entries $p_{st}^{(jk)} \in \mathbb{C}$ determined by*

$$p_{st}^{(jk)} = \sum_{u=1}^{r} \frac{2\zeta_u^{t-s} a_u}{(1 - \beta_j \overline{\zeta_u})^{s+1}(1 - \overline{\beta_k}\zeta_u)^{t+1}}.$$

Corollary 19. *Let $r \in \mathbb{N}$ and let φ be a 1×1 Carathéodory function which is degenerate of order r and which satisfies $\mathfrak{Re}\varphi(w_0) \neq 0$ for a $w_0 \in \mathbb{D}$. Further, let $m \in \mathbb{N}_0$ and $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$ be such that $n = r$, where n is the number defined by* (5), *and let $(\beta_k)_{k=0}^m \in \mathscr{T}_{0,m}^{\#}$. Moreover, let $s := \operatorname{rank} \mathbf{P}_{m,\beta,l}^{(\varphi)}$. Then there are a sequence $(\zeta_u)_{u=1}^s$ of distinct unimodular complex numbers, a sequence $(a_u)_{u=1}^s$ of positive real numbers, and a real number h such that*

$$\varphi(w) = \sum_{u=1}^{s} \frac{\zeta_u + w}{\zeta_u - w} a_u + \mathrm{i}h, \quad w \in \mathbb{D}.$$

As a final remark, the results stated in this paper can be used to prove similar statements on generalized Schwarz–Pick–Potapov block matrices of Schur functions and of functions which belong to the Potapov class.

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